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RESEARCH ARTICLE

When different norms lead to same billiard trajectories?

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Abstract
Extending a result of Milena Radnović and Serge Tabachnikov, we establish conditions for two different non-symmetric norms to define the same billiard reflection law.

Keywords Billiards · Finsler geometry · Hamiltonian systems

Mathematics Subject Classification 53B40 · 53D99 · 70H05

Radnović [6] and independently Tabchnikov [7, Section 2] made the following remarkable observation:

Theorem 1 *Let $\|\cdot\|_\xi$ be a not necessarily symmetric norm in the plane, having an ellipse with focus at the origin o as the unit circle. Then $\|\cdot\|_\xi$ defines the same law of reflection as the Euclidean metric: The angle of reflection equals to the angle of incidence.*

In the first paper it was also noticed that it leads to the fact that billiard trajectories in the plane with norm defined by an ellipse as the unit circle, are the same as in the Euclidean plane after a suitably chosen affine transform.

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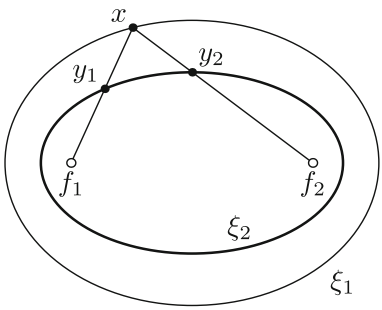

Fig. 1 $\frac{|xf_1|}{|y_1f_1|} + \frac{|xf_2|}{|y_2f_2|} = \text{const}$

Another consequence of this theorem is that the Euclidean and normed ellipses with foci o and f coincide, where f is the second focus of ξ. Indeed, if x is a point in the plane, then by Theorem 1 the differential of $\|\overrightarrow{ox}\|_\xi + \|\overrightarrow{xf}\|_\xi$ is proportional to the differential of the same expression for the Euclidean norm for every x. Hence the value $\|\overrightarrow{ox}\|_\xi + \|\overrightarrow{xf}\|_\xi$ does not change when x moves along the ellipse with foci f and o, in the zero direction of both differentials.

It is interesting, that the latter statement may be deciphered to the following elementary geometric formulation, for which we do not know any short synthetic proof essentially different from the one stated in the above paragraph.

Corollary 2 *Let ξ_1 and ξ_2 be two confocal ellipses with foci at f_1 and f_2. For each point x on ξ_1, denote by y_1 and y_2 the points of intersections of ξ_2 with rays f_1x and f_2x respectively* (*Fig. 1*). *Then for any point x on ξ_1,*

$$\frac{|xf_1|}{|y_1f_1|} + \frac{|xf_2|}{|y_2f_2|} = \text{const}.$$

It can be shown, that the constant in the corollary above equals

$$\frac{\ell_1 - |f_1f_2|}{\ell_2 - |f_1f_2|} + \frac{\ell_1 + |f_1f_2|}{\ell_2 + |f_1f_2|},$$

where ℓ_1 and ℓ_2 are the major axes of the ellipses ξ_1 and ξ_2.

Now we extend the Radnović–Tabachnikov theorem to normed spaces in higher dimension.

Theorem 3 *Let K be a smooth convex body in $\mathbb{R}^n$ containing the origin, and T be its convex image under a projective transform, that maps each line passing through origin to itself preserving its orientation at the origin* (*Fig. 2*). *Then the billiard reflection law in the space with norm $\|\cdot\|_K$ is the same as in the space with norm $\|\cdot\|_T$.*

Remark 4 It is known (see [3, Lemma 4.6]) that such kind of projective transforms send spheres with center at origin to ellipsoids with one of the foci at the origin. Therefore this theorem directly generalizes Theorem 1 to higher dimension.

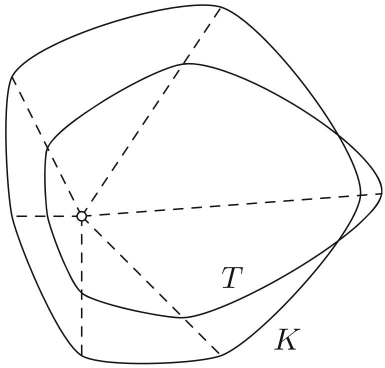

Fig. 2 Convex body K and its projective image T

Remark 5 The law of reflection is not well defined for convex bodies K that are not strictly convex. In this case we may follow the conventions in [1] and define billiard trajectories, for which the reflection direction is not uniquely defined.

Proof We use an idea from [2,4,5], suggesting to work with billiard trajectory in the Banach space $U = \mathbb{R}^n$ with norm $\|\cdot\|_K$ in terms of momenta in the dual space U^* with norm $\|\cdot\|_{K^\circ}$, whose unit ball is the polar body K°. From the smoothness of K, to each unit velocity $u \in \partial K$ there corresponds a conjugate unit momentum $u^* \in \partial K^\circ$ such that $u^*(u) = 1$ and $\|u^*\|_{K^\circ} = 1$. The equation $u^*(x) = 1$ defines the hyperplane tangent to ∂K at u, while the equation $u(y) = 1$ defines the hyperplane tangent to ∂K° at u^*.

Let $q_1q_2q_3$ be a part of a billiard trajectory in U, where q_2 is the point where the trajectory hits a hypersurface S and reflects. Then the sum $\|\overrightarrow{q_1x}\|_K + \|\overrightarrow{xq_3}\|_K$ as a function of $x \in S$ has a critical value at q_2. The criticality in terms of first derivatives means:

$$u_2^* - u_1^* = \lambda n^*, \tag{$*$}$$

where u_1^* and u_2^* are momenta corresponding to unit vectors in directions $\overrightarrow{q_1q_2}$ and $\overrightarrow{q_2q_3}$, and n^* is the normal covector to S at q_2.

It is crucial that equation ($*$) is preserved under a positive similarity of the body K°, possibly with different factor λ. Indeed, let $T^\circ = tK^\circ + v^*$, where $t > 0$ and $v^* \in U^*$. It is easy to see that the momentum, corresponding to velocity u with respect to the body T°, equals $u_T^* = tu^* + v^*$, because u is a linear function on U^* and the points, where its maximum is obtained on K° and T° are moved one to another by the homothety. Hence the difference of the new momenta at a reflection point equals to $t(u_2^* - u_1^*)$, which is still parallel to n^*.

We obtain that the reflection laws for two norms $\|\cdot\|_K$ and $\|\cdot\|_T$ coincide if K° and T° are positive homothets of each other. A positive homothety is a projective transform which maps any point at infinity to itself. Therefore in the dual space (our original U) a positive homothety corresponds to the map, which preserves its polar images as sets, that is planes passing through the origin. It is easy to see that this is the projective transform described in the statement of the theorem.

In simpler words, K is given by the system of linear inequalities of the form

$$u^*(x) \leqslant 1, \quad \text{for all} \quad u^* \in K^\circ.$$

Hence the equations of T must be (assuming working not far from the origin, where $v(x) < 1$)

$$tu^*(x) + v^*(x) \leqslant 1 \quad \iff \quad u^*\left(\frac{tx}{1 - v(x)}\right) \leqslant 1, \quad \text{for all} \quad u^* \in K^\circ.$$

It remains to note that

$$x \mapsto \frac{tx}{1 - v(x)}, \quad t > 0,$$

is the general form of projective maps that preserve lines thorough the origin and keep their orientations at the origin. □

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