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Erdős-Hajnal-type results for monotone paths



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ABSTRACT

An *ordered graph* is a graph with a linear ordering on its vertex set. We prove that for every positive integer k , there exists a constant $c_k > 0$ such that any ordered graph G on n vertices with the property that neither G nor its complement contains an induced monotone path of size k , has either a clique or an independent set of size at least n^{c_k} . This strengthens a result of Bousquet, Lagoutte, and Thomassé, who proved the analogous result for unordered graphs.

A key idea of the above paper was to show that any unordered graph on n vertices that does not contain an induced path of size k , and whose maximum degree is at most $c(k)n$ for some small $c(k) > 0$, contains two disjoint linear size subsets with no edge between them. This approach fails for ordered graphs, because the analogous statement is false for $k \geq 3$, by a construction of Fox. We provide some further examples showing that this statement also fails for ordered graphs avoiding other ordered trees.

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1. Introduction

Erdős and Hajnal [11] proved that graphs avoiding some fixed induced subgraph or subgraphs have very favorable Ramsey-theoretic properties. In particular, they contain surprisingly large homogeneous (that is, complete or empty) subgraphs and bipartite subgraphs. According to the celebrated Erdős-Hajnal conjecture, every graph G on n vertices which does not contain some fixed graph H as an induced subgraph, has a clique or an independent set of size at least n^c , where $c = c(H) > 0$ is a constant that depends only on H . There is a rapidly growing body of literature studying this conjecture (see, e.g., [1,2,5,6,8,12,14,16,24]).

For any graph G and any disjoint subsets $A, B \subset V(G)$, we say that A is complete to B if $ab \in E(G)$ for every $a \in A, b \in B$. If $|A| = |B| = k$ and A is complete to B , then A and B are said to form a *bi-clique of size k* . Denote the maximum degree of the vertices in G by $\Delta(G)$. Following [14], a family of graphs \mathcal{G} is said to have the *Erdős-Hajnal property* if there exists a constant $c = c(\mathcal{G}) > 0$ such that every $G \in \mathcal{G}$ has either a clique or an independent set of size at least $|V(G)|^c$. The family \mathcal{G} has the *strong Erdős-Hajnal property* if there exists a constant $b = b(\mathcal{G}) > 0$ such that for every $G \in \mathcal{G}$, either G or its complement \overline{G} has a bi-clique of size $b|V(G)|$. It was proved in [1] that if a *hereditary* family (that is, a family closed under taking induced subgraphs) has the strong Erdős-Hajnal property, then it also has the Erdős-Hajnal property.

The aim of this paper is to discuss Erdős-Hajnal type problems for ordered graphs. An *ordered graph* is a graph with a total ordering on its vertex set. With a slight abuse of notation, in every ordered graph, we denote this ordering by \prec . If the vertex set of G is a subset of the integers, then \prec stands for the natural ordering. An ordered graph H is an *ordered subgraph* (or simply subgraph) of G if there exists an order preserving embedding from $V(H)$ to $V(G)$ that maps edges to edges. If, in addition, non-edges are mapped into non-edges, then H is called an *induced ordered subgraph* of G . If G does not have H as induced ordered subgraph, then we say that G *avoids* H . The ordered path with vertices $1, \dots, k$ and edges $\{i, i+1\}$, for $i = 1, \dots, k-1$, is called a *monotone path of size k* .

Our main result is the following.

Theorem 1. *For any positive integer k , there exists $c = c(k) > 0$ with the following property. If G is an ordered graph on n vertices such that neither G nor its complement contains an induced monotone path of size k , then G has either a clique or an independent set of size at least n^c .*

One can deduce from our proof that $c(k) = k^{-5-\alpha(1)}$ suffices, but in order to make the paper more readable, we will not include the computations. Our theorem obviously implies the analogous statement for unordered graphs, which was first established by Bousquet, Lagoutte, and Thomassé [5]. The idea of their proof was the following. We call a family of graphs, \mathcal{H} , *lopsided* if there exists a constant $c = c(\mathcal{H}) > 0$ with the

following property: any graph G on n vertices which does not contain any element of \mathcal{H} as an induced subgraph, and for which $\Delta(G) < cn$, the complement of G has a bi-clique of size at least cn . If \mathcal{H} consists of a single graph H , then H is called lopsided. They proved that the (unordered) path of size k is lopsided. It follows from the arguments of Bousquet *et al.* that if \mathcal{H} is lopsided, then the family of all graphs which avoid every element of \mathcal{H} as an induced subgraph, and whose complements also avoid them, has the strong Erdős-Hajnal and, thus, the Erdős-Hajnal property.

Since then, this idea has been exploited to prove the Erdős-Hajnal conjecture for various other families of graphs: the family of graphs avoiding a tree T and its complement [8], the family of graphs avoiding all subdivisions of a graph H and the complements of these subdivisions [9], the family of graphs avoiding a graph H as a vertex minor [7], families of graphs avoiding a fixed cycle as a pivot minor [17], etc.

However, for ordered graphs, this method does not work even in the simplest case: for monotone paths. A construction of Fox [13] shows that, for every n and $\delta > 0$, there exists an ordered graph G with $|V(G)| = n$ and $\Delta(G) < n^\delta$ which avoids the monotone path of size 3, and whose complement does not contain a bi-clique of size larger than $\frac{cn}{\log n}$, for a suitable constant $c = c(\delta) > 0$. Hence, using the above terminology, the monotone path of size at least 3 is *not lopsided*.

Although monotone paths are not lopsided, they satisfy a somewhat weaker property, as is shown by the following theorem of the authors.

Theorem 2. ([20]) *For any positive integer k , there exists a constant $c = c(k) > 0$ with the following property. If G is an ordered graph on n vertices that does not contain an induced monotone path of size k , and $\Delta(G) < cn$, then the complement of G contains a bi-clique of size at least $\frac{cn}{\log n}$.*

Unfortunately, Theorem 1 cannot be deduced from this weaker property. Our approach is based on a technique in [25], where it was shown that the family of string graphs has the Erdős-Hajnal property.

Recently, Seymour, Scott, and Spirkl [24] extended our Theorem 2 from monotone paths to all ordered forests T , albeit with a weaker bound $n^{1-o(1)}$ in place of $\frac{cn}{\log n}$. They proved that for any $0 < c < 1$, there exists $\epsilon = \epsilon(T, c) > 0$ with the following property. If G is an ordered graph on n vertices that does not contain T as an induced ordered subgraph and $\Delta(G) < \epsilon n$, then the complement of G contains a bi-clique of size at least ϵn^{1-c} . Therefore, if we want to guarantee a bi-clique of size $n^{1-o(1)}$ in \overline{G} , we need to assume that the maximum degree of G is $o(n)$. This is definitely a stronger condition than the one we had for monotone paths.

Our next construction shows that this stronger condition is indeed necessary. We also provide new examples of ordered trees T (that do not contain a monotone path of size 3), for which one cannot expect to find linear size bi-cliques.

Theorem 3. *For any $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ with the following property.*

For any positive integer $n \geq n_0$, there is an ordered graph G with n vertices and $\Delta(G) \leq \epsilon n$ such that the size of the largest bi-clique in \overline{G} is at most $n^{1-\delta}$, and G does not contain either of the following ordered trees as an induced ordered subgraph:



The investigation of bipartite variants of the problems considered in this paper were initiated in [18]; see also [3,23].

Our paper is organized as follows. In Section 2, we introduce the key concept needed for the proof of Theorem 1 and reduce Theorem 1 to another statement (Theorem 6). Sections 3 and 4 are devoted to the proof of this latter statement. The construction proving Theorem 3 will be presented in Sections 5.

Throughout this paper, we use the following notation, which is mostly conventional. For any graph G and any subset $U \subset V(G)$, we denote by $G[U]$ the subgraph of G induced by U . The neighborhood of U is defined as $N_G(U) = N(U) = \{v \in V(G) \setminus U : \exists u \in U, uv \in E(G)\}$. If $U = \{u\}$, instead of $N(U)$, we simply write $N(u)$. For a vertex $v \in V(G)$, let $G - v$ stand for the graph obtained from G by deleting the vertex v . Also, if G is an ordered graph, the forward neighborhood of a vertex $v \in V(G)$, denoted by $N_G^+(v) = N^+(v)$ is the set of neighbors y such that $x \prec y$.

For easier readability, we omit the use of floors and ceilings, whenever they are not crucial.

2. The quasi-Erdős-Hajnal property

After introducing some notation and terminology, we outline our proof strategy for Theorem 1.

For any $k \geq 3$, let \mathcal{P}_k denote the family of all ordered graphs G such that neither G nor its complement contains a monotone path of size k as an induced subgraph. Instead of proving that \mathcal{P}_k has the Erdős-Hajnal property, we prove that it has the *quasi-Erdős-Hajnal property*. This concept was introduced by the second named author in [25], in order to show that the family of string graphs has the Erdős-Hajnal property.

Definition 4. A family of graphs, \mathcal{G} , has the *quasi-Erdős-Hajnal property* if there is a constant $c = c(\mathcal{G}) > 0$ with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and t pairwise disjoint subsets $X_1, \dots, X_t \subset V(G)$ such that $t \geq \left(\frac{|V(G)|}{|X_i|}\right)^c$ holds for every $i \in [t]$, and

- (i) either there is no edge between X_i and X_j for $1 \leq i < j \leq t$,
- (ii) or X_i is complete to X_j for $1 \leq i < j \leq t$.

It was proved in [25] that in hereditary families, the quasi-Erdős-Hajnal property is equivalent to the Erdős-Hajnal property. We somewhat relax the definition of the quasi-Erdős-Hajnal property, and with a slight abuse of notation, we overwrite the previous definition as follows.

Definition 5. A family of graphs, \mathcal{G} , has the *quasi-Erdős-Hajnal property* if there are two constants, $\alpha, \beta > 0$, with the following property. For every $G \in \mathcal{G}$ with at least 2 vertices, there exist $t \geq 2$ and t pairwise disjoint subsets $X_1, \dots, X_t \subset V(G)$ such that $t \geq \alpha \left(\frac{|V(G)|}{|X_i|}\right)^\beta$ holds for every $i \in [t]$, and

- (i) either there is no edge between X_i and X_j for $1 \leq i < j \leq t$,
- (ii) or X_i is complete to X_j for $1 \leq i < j \leq t$.

It is easy to verify that the two definitions are in fact equivalent. If \mathcal{G} satisfies Definition 4, then, obviously, it also satisfies Definition 5. In the reverse direction, setting $c = \frac{\beta}{1 - \log_2 \alpha}$ if $\alpha \leq 1$, and $c = \beta$ if $\alpha > 1$, if the inequality $t \geq \alpha \left(\frac{|V(G)|}{|X_i|}\right)^\beta$ holds for some $t \geq 2$, then we also have $t \geq \left(\frac{|V(G)|}{|X_i|}\right)^c$.

Therefore, it is enough to show that \mathcal{P}_k has the quasi-Erdős-Hajnal property. The advantage of the quasi-Erdős-Hajnal property over the Erdős-Hajnal property is that it allows us to establish the following lopsided statement, which will imply Theorem 1.

Theorem 6. *For every positive integer k , there exist two constants $\epsilon, \alpha > 0$ with the following property.*

Let G be an ordered graph on n vertices with maximum degree at most ϵn such that G does not contain a monotone path of size k as an induced subgraph. Then there exist $t \geq 2$ and t pairwise disjoint subsets $X_1, \dots, X_t \subset V(G)$ such that $t \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2}$ holds for every $i \in [t]$, and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

Our proof shows that $\epsilon = 2^{-O(k)}$ and $\alpha = 2^{-O(k)}$ suffice. In the inequality $t \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2}$, the exponent $1/2$ has no significance: the statement remains true with any $0 < \beta < 1$ instead of $1/2$ (with the cost of changing ϵ and α). However, it is not true with $\beta = 1$, as it would contradict the aforementioned construction of Fox [13].

In the rest of this section, we show how Theorem 6 implies Theorem 1. Very similar ideas were used in [5,8,9]. The next two sections are devoted to the proof of Theorem 6.

By a classical result of Rödl [21], any graph G avoiding some fixed graph H contains a linear size subset that is either very dense or very sparse. A quantitatively stronger version of this result was proved by Fox and Sudakov [15].

Lemma 7. [21] *For every graph H and $\epsilon_0 > 0$, there exists $\delta_0 > 0$ with the following property.*

For any graph G with n vertices that does not contain H as an induced subgraph, there is a subset $U \subset V(G)$ such that $|U| \geq \delta_0 n$, and either $|E(G[U])| \leq \epsilon_0 \binom{|U|}{2}$ or $|E(G[U])| \geq (1 - \epsilon_0) \binom{|U|}{2}$.

Lemma 7 applies to unordered graphs, but it can be easily extended to ordered graphs, using the following statement.

Lemma 8. [22] *For every ordered graph H , there exists an unordered graph H_0 with the property that introducing any total ordering on $V(H_0)$, the resulting ordered graph H'_0 always contains H as an induced ordered subgraph.*

By the combination of these two lemmas, we obtain the following.

Lemma 9. *For every ordered graph H and $\epsilon > 0$, there exists $\delta > 0$ with the following property.*

For any ordered graph G with n vertices that does not contain H as an induced ordered subgraph, there exists a subset $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) \leq \epsilon|U|$ or $\Delta(\overline{G}[U]) \leq \epsilon|U|$.

Proof. By Lemma 8, there exists a graph H_0 such that introducing any total ordering on $V(H_0)$, the resulting ordered graph H'_0 contains H as an induced ordered subgraph. Let $\epsilon_0 = \frac{\epsilon}{2}$, and let δ_0 be the constant given by Lemma 7 with respect to H_0 and ϵ_0 .

Let G be an ordered graph with n vertices that does not contain H as an induced ordered subgraph. Then the underlying unordered graph of G does not contain H_0 as an induced subgraph. Hence, there exists $U' \subset V(G)$ such that $|U'| \geq \delta_0 n$, and either $|E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2}$ or $|E(G[U'])| \geq (1 - \epsilon_0) \binom{|U'|}{2}$. Suppose that $|E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2}$, the other case can be handled similarly. Let W be the set of vertices in U' whose degree in $G[U']$ is larger than $2\epsilon_0|U'|$. Then

$$\frac{1}{2}(2\epsilon_0|W|)|U'| \leq |E(G[U'])| \leq \epsilon_0 \binom{|U'|}{2},$$

so that $|W| \leq \frac{|U'|}{2}$. Setting $U = U' \setminus W$, we have $\Delta(G[U]) \leq 2\epsilon_0|U'| \leq \epsilon|U|$ and

$$|U| \geq \frac{|U'|}{2} \geq \frac{\delta_0}{2}n.$$

Hence, $\delta = \frac{\delta_0}{2}$ will suffice. \square

After this preparation, it is easy to deduce from Theorem 6 that \mathcal{P}_k has the quasi-Erdős-Hajnal property and, therefore, the Erdős-Hajnal property.

Proof of Theorem 1. Let $\epsilon, \alpha > 0$ be the constants given by Theorem 6, and let $\delta > 0$ be the constant given by Lemma 9, where H is the monotone path of size k .

Let G be an ordered graph on n vertices such that neither G nor its complement contains a monotone path of length k as an induced subgraph. Then there exists $U \subset V(G)$ such that $|U| \geq \delta n$, and either $\Delta(G[U]) < \epsilon|U|$ or $\Delta(\overline{G}[U]) < \epsilon|U|$. Suppose that $\Delta(G[U]) < \epsilon|U|$, the other case can be handled similarly. Applying Theorem 6 to $G[U]$, we obtain that there exist $t \geq 2$ and t pairwise disjoint sets $X_1, \dots, X_t \subset U$ such that

$$t \geq \alpha \left(\frac{|U|}{|X_i|} \right)^{1/2} \geq \alpha \delta^{1/2} \left(\frac{n}{|X_i|} \right)^{1/2}$$

for $i = 1, \dots, t$, and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

Thus, the family \mathcal{P}_k has the quasi-Erdős-Hajnal property with parameters $\alpha := \alpha \delta^{1/2}$ and $\beta := 1/2$. Therefore, \mathcal{P}_k also has the Erdős-Hajnal property, see Lemma 8. in [25]. \square

In the next two sections, we present the proof of Theorem 6.

3. The main lemma

The backbone of the proof of Theorem 6 is the following technical lemma, whose proof is already contained in [25], within the proof Lemma 7. For convenience and to make this paper self-contained, it is also included here. Recently, our lemma was also utilized by Chudnovsky et al. [10], who provided a different proof as well.

Lemma 10. *There exist two constants $0 < \epsilon, \alpha < \frac{1}{4}$ with the following property. Let H be a bipartite graph with vertex classes A and B , $|A| = |B| = n$. Then at least one of the following three conditions is satisfied.*

- (i) *There exist $t \geq 2$ and $2t$ pairwise disjoint sets $W_1, \dots, W_t \subset A$ and $X_1, \dots, X_t \subset B$ such that $t \geq \alpha \left(\frac{n}{|X_i|} \right)^{1/2}$, and $X_i \subset N(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N(W_j) = \emptyset$ for $i \neq j$.*
- (ii) *There exist $X \subset A$ and $Y \subset B$ such that $|X|, |Y| > \frac{n}{4}$, and there is no edge between X and Y .*
- (iii) *There exists $v \in A$ such that $|N(v)| \geq \epsilon n$.*

Let us briefly outline the idea of the proof. We want to find an induced subgraph H' of H with vertex classes $A' \subset A$ and $B' \subset B$ such that

- H' is almost bi-regular, more precisely, the degree of every vertex in A' is at most Δ , and the degree of every vertex in B' is within a constant factor of some d ,
- $|A'|$ and $|B'|$ are large with respect to d and Δ .

If we can find such an H' , we construct our sets W_1, \dots, W_t and X_1, \dots, X_t as follows. By a probabilistic argument, we find $S \subset A'$ such that $\Omega(|B'|)$ vertices in B' have exactly

one neighbor in S . Then we can group the vertices in S into sets W_1, \dots, W_t such that $X_i := N_{H'}(W_i)$ has size roughly Δ for $i \in [t]$. The $2t$ sets W_1, \dots, W_t and X_1, \dots, X_t will satisfy (i). We find a suitable H' algorithmically: we either found our desired A' or B' , or there are too few vertices with too large degrees, in which case we remove these vertices and continue. We show that if we cannot find H' , then at least one of (ii) or (iii) must hold.

Proof of Lemma 10. We show that $\epsilon = \frac{1}{2000}$ and $\alpha = \frac{1}{200}$ meet the above requirements.

Suppose that (iii) does not hold. Then the number of edges of H is at most ϵn^2 , so the number of vertices $w \in B$ such that $|N(w)| > \epsilon n$ is at most $n/2$. Deleting all such vertices, and some more, we obtain a bipartite graph H' with vertex classes A' and B' of size $n' = n/2$ such that the maximum degree of H' is at most $2\epsilon n = 4\epsilon n'$.

Let $\epsilon' = 4\epsilon = \frac{1}{500}$ and $\alpha' = \frac{1}{100}$. From now on, we shall only work with H' , so with a slight abuse of notation, write $H := H'$, $A_0 := A'$, $B_0 := B'$, $n := n'$, $\epsilon := \epsilon'$ and $\alpha := \alpha'$. Therefore, we have $\Delta(H) \leq \epsilon n$.

In what follows, we describe an algorithm, which will be referred to as the *main algorithm*. It will output

- (i)' either an integer $t \geq 2$ and $2t$ pairwise disjoint sets $W_1, \dots, W_t \subset A$ and $X_1, \dots, X_t \subset B$ such that $t \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2}$, and $X_i \subset N(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N(W_j) = \emptyset$ for $i \neq j$;
- (ii)' or two subsets $X \subset A$ and $Y \subset B$ such that $|X|, |Y| > \frac{n}{2}$ and there is no edge between X and Y .

We declare the following constants for the main algorithm. Let $J_0 := \lceil \log_2 \epsilon n \rceil + 1$, and for $j = 1, \dots, J_0$, let $t_j := n^{1/2} 2^{j/2}$. Then

$$\sum_{i=1}^{J_0} t_i = \sum_{i=1}^{J_0} n^{1/2} 2^{i/2} \leq 2n\epsilon^{1/2} \frac{1}{1 - 2^{-1/2}} < \frac{n}{4}. \tag{1}$$

Also, let $A_0^* := \emptyset$ and $B_0^* := \emptyset$.

In the q -th step of the main algorithm, we define $A_q, A_q^*, B_q, B_q^*, J_q$. We will think of A_q^* and B_q^* as a set of “leftovers”. That is, we get A_q and A_q^* by transferring certain elements from A_{q-1} to A_{q-1}^* , and we get B_q and B_q^* by transferring certain elements from B_{q-1} to B_{q-1}^* . Also, J_q will keep track of the maximum degree in B_q , and it will decrease after each step. We make sure that the following properties are satisfied:

1. A_q, A_q^*, B_q, B_q^* are pairwise disjoint and $A_q \cup A_q^* = A, B_q \cup B_q^* = B$,
2. $|A_q^*|, |B_q^*| \leq 2 \sum_{i=J_q+1}^{J_0} t_i$,
3. for every $v \in B_q, |N(v) \cap A_q| < 2^{J_q}$.

Note that by (1) and conditions 1 and 2, we have $|A_q|, |B_q| \geq \frac{n}{2}$. Also, the conditions 1-3 are certainly satisfied for $q = 0$. Next, we describe the q -th step of our main algorithm.

Main algorithm. If $J_{q-1} = 0$, then stop the main algorithm, and output $X := A_{q-1}, Y := B_{q-1}$. In this case, there is no edge between A_{q-1} and B_{q-1} by condition 3, and $|A_{q-1}|, |B_{q-1}| \geq \frac{n}{2}$ by condition 2. This output satisfies condition (ii)'.
 Suppose next that $J_{q-1} \geq 1$. For $i = 1, \dots, J_{q-1}$, let V_i be the set of vertices $v \in B_{q-1}$ such that $2^{i-1} \leq |N(v) \cap A_{q-1}| < 2^i$, and let V_0 be the set of vertices $v \in B_{q-1}$ such that $N(v) \cap A_{q-1} = \emptyset$. Then, by condition 3, we have $B_{q-1} = \bigcup_{i=0}^{J_{q-1}} V_i$.

Let $1 \leq k \leq J_{q-1}$ be the largest integer for which $t_k < |V_k|$. First, consider the case where there is no such k . Then

$$n - \sum_{i=J_{q-1}+1}^{J_0} t_i - |V_0| \leq n - |B_{q-1}^*| - |V_0| = |B_{q-1}| - |V_0| = \sum_{i=1}^{J_{q-1}} |V_i| \leq \sum_{i=1}^{J_{q-1}} t_i,$$

where the first inequality follows from condition 2, and the first equality is the consequence of condition 1. Comparing the left-hand and right-hand sides, and using (1), we get $|V_0| \geq n/2$. In this case, stop the algorithm and output $X := A_{q-1}$ and $Y := V_0$. This output satisfies condition (ii)'.

Suppose that there exists k with the desired property. Let $B_{q,0} = B_{q-1} \setminus (\bigcup_{i=k+1}^{J_{q-1}} V_i)$, and let $B_{q,0}^* = B_{q-1}^* \cup (\bigcup_{i=k+1}^{J_{q-1}} V_i)$. Then $|B_{q,0}^*| \leq |B_{q-1}^*| + \sum_{i=k+1}^{J_{q-1}} t_i$ holds. Also, set $J'_q := k, A_{q,0} = A_{q-1}$ and $A_{q,0}^* = A_{q-1}^*$. Note that properties 1-3 are satisfied with $A_{q,0}, A_{q,0}^*, B_{q,0}, B_{q,0}^*, J'_q$ instead of $A_q, A_q^*, B_q, B_q^*, J_q$, respectively.

Now we shall run a *sub-algorithm*. Let $Z_0 = V_k$. With help of the sub-algorithm, we construct a sequence $Z_0 \supset \dots \supset Z_r$ satisfying the following properties. During the ℓ -th step of the sub-algorithm, we either find an output satisfying (i)', or we will transfer certain elements of $A_{q,\ell-1}$ to $A_{q,\ell-1}^*$, resulting in the sets $A_{q,\ell}$ and $A_{q,\ell}^*$. At the end of the ℓ -th step of this algorithm, Z_ℓ will be the set of vertices in $B_{q,0}$ that still have at least 2^{k-1} neighbors in A . We stop the algorithm if Z_ℓ is too small. Let us describe the ℓ -th step of the algorithm.

Sub-algorithm. Suppose that $Z_{\ell-1}, A_{q,\ell-1}, A_{q,\ell-1}^*$ have already been defined. If $|Z_{\ell-1}| < 2t_k$, then let $r = \ell - 1$, stop the sub-algorithm. Set $B_q := B_{q,0} \setminus Z_{\ell-1}, B_q^* := B_{q,0}^* \cup Z_{\ell-1}, A_q := A_{q,\ell-1}, A_q^* := A_{q,\ell-1}^*$, and $J_q := k - 1$. Move to the next step of the main algorithm. Note that conditions 1 and 3 are satisfied, and B_q^* satisfies condition 2. Later, we will see that A_q^* satisfies 2. as well.

On the other hand, if $|Z_{\ell-1}| \geq 2t_k$, we define Z_ℓ as follows. Let $x_\ell = \frac{|Z_{\ell-1}|}{t_k}$. Say that a vertex $v \in A_{q,\ell-1}$ is *heavy* if

$$|N(v) \cap Z_{\ell-1}| \geq \frac{x_{\ell-1} 2^k}{t_k} |Z_{\ell-1}| = \left(\frac{|Z_{\ell-1}|}{t_k} \right)^2 2^k = \frac{|Z_{\ell-1}|^2}{n} =: \Delta_\ell,$$

and let K_ℓ be the set of heavy vertices. Counting the number of edges f between K_ℓ and $Z_{\ell-1}$ in two ways, we can write

$$|K_\ell| \cdot \Delta_\ell \leq f < |Z_{\ell-1}| \cdot 2^k,$$

which gives

$$|K_\ell| < \frac{|Z_{\ell-1}| \cdot 2^k}{\Delta_\ell} = \frac{t_k}{x_\ell},$$

where the equality holds by the definition of Δ_ℓ . Set $A_{q,\ell} := A_{q,\ell-1} \setminus K_\ell$ and $A_{q,\ell}^* = A_{q,\ell-1}^* \cup K_\ell$. Examine how the degrees of the vertices in $Z_{\ell-1}$ changed, and consider the following two cases:

Case 1. At least $\frac{|Z_{\ell-1}|}{2}$ vertices in $Z_{\ell-1}$ have at least 2^{k-1} neighbors in $A_{q,\ell}$.

Let T be the set of vertices in $Z_{\ell-1}$ that have at least 2^{k-1} neighbors in $A_{q,\ell}$, so $|T| \geq \frac{|Z_{\ell-1}|}{2}$. Pick each element of $A_{q,\ell}$ with probability $p = 2^{-k}$, and let S be the set of selected vertices. We say that $v \in T$ is *good* if $|N(v) \cap S| = 1$, and let D be the set of good vertices. We have

$$\mathbb{P}(v \text{ is good}) = |N(v) \cap A_{q,\ell}| p(1-p)^{|N(v) \cap A_{q,\ell}|-1} \geq \frac{1}{2}(1-2^{-k})^{2^k} \geq \frac{1}{6},$$

so that $\mathbb{E}(|D|) \geq \frac{|T|}{6} \geq \frac{|Z_{\ell-1}|}{12}$. Therefore, there exists a choice for S such that $|D| \geq \frac{|Z_{\ell-1}|}{12}$. Let us fix such an S . For each $v \in S$, let D_v be the set of elements $w \in D$ such that $N(w) \cap S = \{v\}$. Also, note that

$$|D_v| \leq |N(v) \cap Z_\ell| \leq \min\{\epsilon n, \Delta_\ell\} =: \Delta'_\ell.$$

In other words, the sets D_v for $v \in S$ partition D into sets of size at most Δ'_ℓ . Here, we have

$$\frac{|D|}{\Delta'_\ell} \geq \frac{|Z_{\ell-1}|}{12\Delta'_\ell} \geq \max\left\{\frac{n}{12|Z_{\ell-1}|}, \frac{|Z_{\ell-1}|}{\epsilon n}\right\}.$$

By the choice of ϵ , the right-hand side is always at least 6. But then we can partition S into $t \geq \frac{|D|}{3\Delta'_\ell} \geq 2$ parts W_1, \dots, W_t such that the sets $X_i = \bigcup_{v \in W_i} D_v$ have size at least Δ'_ℓ for $i = 1, \dots, t$. The integer t and the resulting sets X_1, \dots, X_t satisfy that

$$t \geq \frac{|D|}{3\Delta'_\ell} \geq \frac{n}{36|Z_{\ell-1}|} \geq \frac{1}{36} \left(\frac{n}{\Delta_\ell}\right)^{1/2} \geq \frac{1}{36} \left(\frac{n}{|X_i|}\right)^{1/2}.$$

Stop the main algorithm, and output t and the $2t$ pairwise disjoint sets W_1, \dots, W_t and X_1, \dots, X_t . By the choice of α , this output satisfies (i)?

Case 2. At most $\frac{|Z_{\ell-1}|}{2}$ vertices in $Z_{\ell-1}$ have at least 2^{k-1} neighbors in $A_{q,\ell}$.

In this case, define Z_ℓ as the set of elements of $Z_{\ell-1}$ with at least 2^{k-1} neighbors in $A_{q,\ell}$ (then Z_ℓ is the set of all elements in $B_{q,0}$ with at least 2^{k-1} neighbors in $A_{q,\ell}$ as well). Also, move to the next step of the sub-algorithm.

We show that conditions 1-3 are still satisfied for $A_{q,\ell}, A_{q,\ell}^*, B_{\ell,0}, B_{\ell,0}^*, J'_q$ instead of $A_q, A_q^*, B_q, B_q^*, J_q$. Conditions 1 and 3 are clearly true, and 2 holds for $B_{q,0}^*$. It remains to show that 2 holds for $A_{q,\ell}^*$ as well. Note that, as $|Z_j| \leq \frac{|Z_{j-1}|}{2}$ for $j = 1, \dots, \ell$, and $|Z_{\ell-1}| \geq 2t_k$, we have $|Z_j| \geq 2^{\ell-j}t_k$, and $x_j \geq 2^{\ell+1-j}$. Therefore,

$$\begin{aligned} |A_{q,\ell}^*| &= |A_{q-1}^*| + \sum_{j=1}^{\ell} |K_j| \leq |A_{q-1}^*| + \sum_{j=1}^{\ell} \frac{t_k}{x_j} \leq |A_{q-1}^*| + \sum_{j=1}^{\ell} \frac{t_k}{2^{\ell+1-j}} \\ &< |A_{q-1}^*| + t_k. \end{aligned}$$

Hence, condition 2 is also satisfied.

As we have $J_0 > J_1 > \dots \geq 0$, the main algorithm will stop after at most J_0 steps. When the algorithm stops, its output will satisfy either (i)' or (ii)'. \square

Let us remark that if (i) holds, then the $2t$ sets W_1, \dots, W_t and X_1, \dots, X_t have the additional property that every vertex in $\bigcup_{i=1}^t X_i$ has exactly one neighbor in $\bigcup_{i=1}^t W_i$.

4. The proof of Theorem 6

Now we are in a position to prove Theorem 6. Let G be an ordered graph. The *transitive closure* of G is the ordered graph G' on the vertex set $V(G)$ in which x and y are connected by an edge if and only if there exists a monotone path in G with endpoints x and y .

Let us briefly outline the proof idea. We assume that for G there is no integer $t \geq 2$ and t sets X_1, \dots, X_t with the desired properties. Then we show that G contains a monotone path x_1, \dots, x_k with the following additional property. For $s = 1, \dots, k$, there are $\Omega(n)$ vertices in G that can be reached by a monotone path from x_s , which avoids all the neighbors of x_1, \dots, x_{s-1} . This additional property lets us do induction on s , allowing us to find x_1, \dots, x_k one-by-one.

Proof of Theorem 6. Let $0 < \epsilon_1, \alpha_1 < \frac{1}{4}$ be the constants given by Lemma 10 as ϵ, α , respectively. Furthermore, define the following constants: $c_1 = \frac{\epsilon_1}{2}$, $c_{i+1} = \frac{\epsilon_1 c_i}{4}$ (for $i = 1, 2, \dots$), $\epsilon = \frac{c_k}{2}$, and $\alpha = \frac{\alpha_1 c_k^{1/2}}{2}$.

Let G be an ordered graph on n vertices such that

1. the maximum degree of G is at most ϵn ,
2. there exists no $t \geq 2$ such that for some pairwise disjoint sets $X_1, \dots, X_t \subset V(G)$ we have $t \geq \alpha \left(\frac{n}{|X_i|}\right)^{1/2}$ and there is no edge between X_i and X_j for $1 \leq i < j \leq t$.

Then we show that G contains a monotone path of size k as an induced subgraph. In particular, we find k vertices $x_1 \prec \dots \prec x_k$ with the following properties. For $s = 1, \dots, k$,

- (a) x_1, \dots, x_s form an induced monotone path.
- (b) Let

$$U_s = V(G) \setminus \left(\bigcup_{i=1}^{s-1} N(x_i) \right),$$

let $G_s = G[U_s \cup \{x_s\}]$, and let G'_s be the transitive closure of G_s . Then the forward degree of x_s in G'_s is at least $c_s n$.

First, we find a vertex x_1 with the desired properties, that is, if G' is the transitive closure of G , then the forward degree of x_1 must be at least $c_1 n$. Let A_0 be the set of the first $n/2$ elements of $V(G)$, and set $B_0 = V(G) \setminus A_0$. Also, let H_0 denote the bipartite subgraph of G' with parts A_0 and B_0 . By Lemma 10, at least one of the following three conditions is satisfied.

- (i) There exist $t \geq 2$ and $2t$ pairwise disjoint sets $W_1, \dots, W_t \subset A_0$ and $X_1, \dots, X_t \subset B_0$ such that

$$t \geq \alpha_1 \left(\frac{|A_0|}{|X_i|} \right)^{1/2} = 2^{-1/2} \alpha_1 \left(\frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|} \right)^{1/2},$$

and $X_i \subset N_{H_0}(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N_{H_0}(W_j) = \emptyset$ for $i \neq j$.

- (ii) There exist $X \subset A_0$ and $Y \subset B_0$ such that $|X|, |Y| \geq \frac{n}{8}$, and there is no edge between X and Y .
- (iii) There exists $v \in A_0$ such that $|N_{H_0}(v)| \geq \epsilon_1 |A_0| = c_1 n$.

As the non-edges of G' are also non-edges of G , (ii) cannot hold. Otherwise, $t = 2$ and $X_1 = X, X_2 = Y$ contradicts property 2 of G . Suppose that (i) holds. Note that there is no edge between X_i and X_j in G , for $1 \leq i < j \leq t$. Suppose for contradiction that $x \in X_i$ and $y \in X_j$ are joined by an edge in G , for some $x \prec y$. Then there exists $w \in W_i$ such that $wx \in E(G')$, but $wy \notin E(G')$. This is a contradiction, as this means that there is a monotone path from w to x in G , so there is a monotone path from w to y as well. Hence, there is no edge between X_i and X_j for $1 \leq i < j \leq t$, which contradicts 2. Therefore, (iii) must hold: there exists a vertex $x_1 \in V(G)$ whose forward degree in $G' = G'_1$ is at least $c_1 n$.

Suppose that we have already found x_1, \dots, x_s with the desired properties, for some $1 \leq s \leq k - 1$. Then we define x_{s+1} as follows. Let F_s be the forward neighborhood of x_s in G_s , let K_s be the forward neighborhood of x_s in G'_s , and let $L_s = K_s \setminus F_s$. As $|F_s| \leq \epsilon n$ and $|K_s| \geq c_s n$, we have $|L_s| \geq \frac{c_s}{2} n$. Let A_s be the set of the first $\frac{|L_s|}{2}$ elements of L_s with respect to \prec , and let $B_s = L_s \setminus A_s$. A monotone path in G_s is said to be *good* if none of its vertices, with the possible exception of the first one, belongs to F_s . For every $v \in A_s$, there exists at least one element $x \in F_s$ such that $v \in N_{G'_s}^+(x)$; assign the largest (with respect to \prec) such element x to v . Then there is a good monotone path from x to v .

Define a bipartite graph H_s between A_s and B_s as follows. If $v \in A_s$ and $y \in B_s$, and $x \in F_s$ is the vertex assigned to v , then join v and y by an edge if there is a good monotone path from x to y . Applying Lemma 10 to H_s , we conclude that at least one of the following three statements is true.

- (i) There exist $t \geq 2$ and $2t$ pairwise disjoint sets $W_1, \dots, W_t \subset A_s$ and $X_1, \dots, X_t \subset B_s$ such that

$$t \geq \alpha_1 \left(\frac{|A_s|}{|X_i|} \right)^{1/2} > \frac{\alpha_1 c_s^{1/2}}{2} \left(\frac{n}{|X_i|} \right)^{1/2} \geq \alpha \left(\frac{n}{|X_i|} \right)^{1/2},$$

and $X_i \subset N_{H_s}(W_i)$ for $i = 1, \dots, t$, but $X_i \cap N_{H_s}(W_j) = \emptyset$ for $i \neq j$.

- (ii) There exist $X \subset A_s$ and $Y \subset B_s$ such that $|X|, |Y| \geq \frac{|A_s|}{4} \geq \frac{c_s n}{16}$, and there is no edge between X and Y in H_s .
- (iii) There exists $v \in A_s$ such that $|N_{H_s}(v)| \geq \epsilon_1 |A_s| = \frac{\epsilon_1 c_s}{4} n = c_{s+1} n$.

Suppose first that (i) holds. Then, as before, we show that there is no edge between X_i and X_j in G for $1 \leq i < j \leq t$. Suppose that $u \in X_i$ and $w \in X_j$ are joined by an edge in G , for some $u \prec w$. Then there exists $v \in W_i$ such that $vu \in E(H_s)$, but $vw \notin E(H_s)$. Let $x \in F_s$ be the vertex assigned to v . Then we can find a good monotone path from x to u . Since uw is an edge of G , there is a good monotone path from x to w , contradicting the assumption $vw \notin E(H_s)$. Therefore, there cannot be any edge between X_i and X_j in G , which means that (i) contradicts 2.

Suppose next that (ii) holds. Again, we can show that there is no edge between X and Y in G , which then contradicts 2. by setting $t = 2$ and $X_1 = X, Y_1 = Y$. Suppose that $v \in X$ and $y \in Y$ are joined by an edge in G , and let $x \in F_s$ be the vertex assigned to v . There is a good monotone path from x to v in G_s , so there is a good monotone path from x to y , contradicting the assumption that vy is not an edge of H_s .

Therefore, we can assume that (iii) holds. Let $v \in A_s$ be a vertex of degree at least $c_{s+1} n$ in H_s , and let $x_{s+1} \in F_s$ be the vertex assigned to v . We show that x_{s+1} satisfies the desired properties. We have $U_{s+1} = U_s \setminus F_s$, and the forward degree of x_{s+1} in G'_{s+1} is exactly the number of vertices y such that there is a good monotone path from x_{s+1} to y . That is, the forward degree of x_{s+1} is at least $|N_{H_s}(v)| \geq c_{s+1} n$, as required. This completes the proof. \square

5. The construction—Proof of Theorem 3

In this section, we present our construction for Theorem 3. The construction involves expander graphs, which are defined as follows.

The *closed neighborhood* of U in a graph H is defined as $U \cup N_H(U)$, and is denoted by $N[U] = N_H[U]$. The graph H is called an (n, d, λ) -*expander* if H is a d -regular graph on n vertices, and for every $U \subseteq V$ satisfying $|U| \leq |V|/2$, we have $|N_H[U]| \geq (1 + \lambda)|U|$. By a well-known result of Bollobás [4], a random 3-regular graph on n vertices is a $(n, 3, \lambda_0)$ -expander with high probability for some absolute constant $\lambda_0 > 0$. In the rest of this section, we fix such a constant λ_0 . For explicit constructions of expander graphs see, e.g., [19].

For any positive integer r , let H^r denote the graph with vertex set $V(H)$ in which two vertices are joined by an edge if there exists a path of length at most r between them in H . Here we allow loops, so that in H^r every vertex is joined to itself. We need the following simple property of expander graphs.

Claim 11. *Let H be an (n, d, λ) -expander graph and let $r \geq 1$. For any subsets $X, Y \subseteq V(H)$ such that there is no edge between X and Y in H^r , we have $|X| \cdot |Y| \leq n^2(1 + \lambda)^{-r}$.*

Proof. Let $X_i = N_{H^i}[X]$ and $Y_i = N_{H^i}[Y]$ for $i = 1, \dots, r$, and let $X_0 = X, Y_0 = Y$. It follows from the definition of expanders that, if $|X_i| \leq \frac{n}{2}$, then

$$|X| \leq \frac{1}{2}n(1 + \lambda)^{-i}.$$

Similarly, if $|Y_i| \leq \frac{n}{2}$, then $|Y| \leq \frac{1}{2}n(1 + \lambda)^{-i}$. If X and Y are not connected by any edge in H^r , then X_i and Y_{r-i} must be disjoint for every i . Let ℓ be the largest number in $\{0, 1, \dots, r\}$ such that $|X_\ell| \leq n/2$.

If $\ell = r$, then $|X| < n(1 + \lambda)^{-r}$, and hence $|X||Y| \leq n^2(1 + \lambda)^{-r}$.

If $\ell < r$, then $|X_{\ell+1}| > n/2$ and $|Y_{r-\ell-1}| \leq n/2$. Therefore, we have $|Y| \leq n(1 + \lambda)^{-(r-\ell-1)}$. Using the inequality $1 + \lambda \leq 2$, we obtain

$$|X| \cdot |Y| \leq \frac{1}{4}n^2(1 + \lambda)^{-r+1} \leq n^2(1 + \lambda)^{-r}. \quad \square$$

Claim 12. *For any d -regular graph H and $r \geq 1$, we have $\Delta(H^r) \leq (d + 1)^r$.*

Our construction is based on the following key lemma.

Lemma 13. *Let k, m, f be positive integers. Let A_1, \dots, A_k be pairwise disjoint sets of size m , and suppose that there exists an $(m, 3, \lambda_0)$ -expander.*

Then there is a graph G on the vertex set $V = \bigcup_{i=1}^k A_i$ such that

1. $\Delta(G) \leq 4^{f2^k}$;

2. if $x, y, z \in V$ such that $x \in A_a, y \in A_b, z \in A_c$ for some $a < b < c$, and $xy, xz \in E(G)$, then $yz \in E(G)$ as well;
3. for any $a \neq b$ and any pair of subsets $X \subset A_a$ and $Y \subset A_b$ that are not connected by any edge of G , we have $|X| \cdot |Y| \leq m^2(1 + \lambda_0)^{-f}$.

Proof. Let H be an $(m, 3, \lambda_0)$ -expander. Let $\phi : V \rightarrow V(H)$ be an arbitrary function such that ϕ is a bijection when restricted to the set A_i , for $i = 1, \dots, k$. Define the graph G , as follows. Suppose that $x \in A_a$ and $y \in A_b$ for some $a < b$. Join x and y by an edge if there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(y)$ in H . By Claim 12, the maximum degree of G is at most $\sum_{i=1}^{k-1} 4f2^i \leq 4f2^k$, so that G has property 1.

To see that G also has property 2, consider $x \in A_a, y \in A_b, z \in A_c$ such that $a < b < c$ and $xy, xz \in E(G)$. We have to show that $yz \in E(G)$. By definition, there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(y)$ in H , and there exists a path of length at most $f2^{a-1}$ between $\phi(x)$ and $\phi(z)$. But then there exists a path of length at most $f2^a \leq f2^{b-1}$ between $\phi(y)$ and $\phi(z)$, so yz is also an edge of G .

It remains to verify that G has property 3. If $1 \leq a < b \leq k$ and $X \subset A_a$ and $Y \subset A_b$ are not connected by any edge in G , then there is no edge between $\phi(X)$ and $\phi(Y)$ in $H^{f2^{a-1}}$. By Claim 11, we have $|X| \cdot |Y| \leq m^2(1 + \lambda_0)^{-f2^{a-1}} \leq m^2(1 + \lambda_0)^{-f}$. \square

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. Let $k = \frac{2}{\epsilon}, f = \frac{\log_2 n}{2 \cdot 2^k}$, and $m = \frac{n}{k}$. We show that the theorem holds with $\delta = \frac{\log_2(1 + \lambda_0)}{2^k}$.

Let A_1, \dots, A_k be pairwise disjoint sets of size m . By Lemma 13, there exists a graph G_0 on $V = \bigcup_{i=1}^m A_i$ satisfying conditions 1-3 with the above parameters.

Define the ordered graph G on the vertex set V as follows. Let \prec be any ordering on V satisfying $A_1 \prec \dots \prec A_k$. For any $x \in A_a$ and $y \in A_b$, join x and y by an edge of G if $xy \in E(G_0)$, or $a = b$. Then the maximum degree of G is at most $\frac{n}{k} + \Delta(G_0) \leq \epsilon n$. Notice that the complement of G does not contain a bi-clique of size $n^{1-\delta}$. Indeed, if (X, Y) is a bi-clique in \overline{G} , then there exists $a \neq b$ such that $|X \cap A_a| \geq \frac{|X|}{k}$ and $|Y \cap A_b| \geq \frac{|Y|}{k} = \frac{|X|}{k}$. Thus,

$$\frac{|X|^2}{k^2} \leq |X \cap A_a| \cdot |Y \cap A_b| \leq m^2(1 + \lambda_0)^{-f} = \frac{m^2}{n^{2\delta}},$$

which implies that $|X| \leq n^{1-\delta}$.

It remains to show that G contains neither S , nor P as an induced ordered subgraph. Let us start with S . Suppose that there are four vertices, $v_0 \prec v_1 \prec v_2 \prec v_3$, in G such that $v_0v_1, v_0v_2, v_0v_3 \in E(G)$, but $v_1v_2, v_2v_3, v_1v_3 \notin E(G)$. Let $v_0 \in A_a, v_1 \in A_b, v_2 \in A_c$, and $v_3 \in A_d$, then $a \leq b \leq c \leq d$. If $c = a$, then $b = a$, which implies $v_1v_2 \in E(G)$, contradiction. Therefore, $a < c \leq d$. As $v_2v_3 \notin E(G)$, we must have $c < d$ as well. But then the three vertices v_0, v_2, v_3 contradict property 2, so that G does not contain S an induced ordered subgraph.

To show that G does not contain P , we can proceed in a similar manner. Suppose for contradiction that there are four vertices, $v_0 \prec v_1 \prec v_2 \prec v_3$, in G such that $v_0v_2, v_0v_3, v_1v_2 \in E(G)$, but $v_0v_1, v_1v_3, v_2v_3 \notin E(G)$. Let $v_0 \in A_a, v_1 \in A_b, v_2 \in A_c$, and $v_3 \in A_d$, where $a \leq b \leq c \leq d$. We have $a < b$, otherwise $v_0v_1 \in E(G)$. In the same way, $c < d$, otherwise $v_2v_3 \in E(G)$. Therefore, $a < c < d$, and the vertices, v_0, v_2 , and v_3 , contradict condition 2 of Lemma 13. \square

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